Portefeuilles efficients au sens de Markowitz — Page 9 slides

↳ Démonstration que (E) est équivalent à (E')

Soit A un ptf solution de (E) tel que $Var[V_1(a_0^A, a^A)] < \sigma^2$

Soit $\delta = \sqrt{\frac{\sigma^2}{Var V_1^\alpha}}$ et considérons le ptf B : $(a_0^B, \delta a^A)$ avec a_0^B choisi tq $V_0^B = v$

$\delta = 1 + \varepsilon \quad , \quad \varepsilon > 0$

$E V_1^A = (v - a' p_0)(1+r) + a' diag(p_0) \mu$

$E V_1^B = (v - \delta a' p_0)(1+r) + \delta a' diag(p_0) \mu$

$= E V_1^A + \varepsilon (a' diag\, p_0 \mu - a' p_0 (1+r))$

$= E V_1^A - v(1+r)$

$Var\, V_1^B = Var\left(a_0^B (1+r) + \delta \sum_{i=1}^N a_i^A p_{i,0} y_i\right)$

$= \delta^2 \left(Var\left(\sum_{i=1}^N a_i^A p_{i,0} y_i \right)\right)$

$= \delta^2 \left(Var\left(a_0^A (1+r) + \sum_{i=1}^N a_i^A p_{i,0} y_i\right)\right)$

$= \delta^2 \, VaR\, V_1^A = \sigma^2$

Résoudre (E') — Slide 11

$\max (w_a' \tilde{\mu})$

s.c $w_a' \Omega w_a - \sigma^2 = 0$

↙ on enlève $v(1+r)$ car constant ne sert à rien dans l'optimisation

forme quadratique :

$w_a' \Omega w_a = \sum_{i=1}^n \sum_{j=1}^n w_{a,i} \Omega_{i,j} w_{a,j}$

$= \sum_{i=1}^n w_{a,i}^2 \Omega_{i,i} + \sum_{i=1}^n \sum_{j \neq i} w_{a,j} \Omega_{i,i} w_{a,j}$

On introduit le lagrangien

$\mathcal{L}(w_a, \lambda) = w_a' \tilde{\mu} - \frac{\lambda}{2}(w_a' \Omega w_a - \sigma^2)$

$= \sum_{i=1}^N w_{a,i} \tilde{\mu}_i - \frac{\lambda}{2}\left(\sum_{i=1}^N \left(w_{a,i}^2 \Omega_{i,i} + \sum_{j \neq i} w_{a,i} w_{a,j} \Omega_{i,j} \right) - \sigma^2 \right)$

$\frac{\partial \mathcal{L}(w_a, \lambda)}{\partial w_{a,i}} = \tilde{\mu}_i - \frac{\lambda}{2}\left(2 w_{a,i} \Omega_{i,i} + 2 \sum_{j \neq i} w_{a,j} \Omega_{i,j} \right)$

$\frac{\partial \mathcal{L}(w_a, \lambda)}{\partial \lambda} = \frac{1}{2}(w_a' \Omega w_a - \sigma^2)$

$$\frac{\partial \mathcal{L}(w_a,\lambda)}{\partial w_a}=0 \qquad \frac{\partial \mathcal{L}(w_a,\lambda)}{\partial \lambda}=0 \iff \begin{cases} \tilde{\mu}-\lambda\Omega w_a=0 \\ w_a'\Omega w_a-\sigma^2=0 \end{cases}$$

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slide 20

On pose $\Omega=R^2$ et $R=\Omega^{1/2}$

Soit (a_0,a) un ptf quelconque

$$S(a)=\frac{w_a'\tilde{\mu}}{\sqrt{w_a'\Omega w_a}}=\frac{w_a'\Omega^{1/2}\Omega^{-1/2}\tilde{\mu}}{\sqrt{w_a'\Omega w_a}}=\frac{(\Omega^{1/2}w_a)'(\Omega^{-1/2}\tilde{\mu})}{\sqrt{(\Omega^{1/2}w_a)'(\Omega^{1/2}w_a)}}=\frac{Tr(YX')}{\sqrt{Tr(XX')}}$$

On pose $X:=\Omega^{1/2}w_a\in\mathbb{R}^n$ et $Y:=\Omega^{-1/2}\tilde{\mu}\in\mathbb{R}^n$

Cauchy-Swartz

$$Tr(YX')\leq\sqrt{Tr(YY')}\sqrt{Tr(X,X')}$$

$$\leq\sqrt{Tr(Y,Y')}$$

$$\leq\sqrt{(\Omega^{-1/2}\tilde{\mu})'(\Omega^{-1/2}\tilde{\mu})}=(\tilde{\mu}'\Omega^{-1}\tilde{\mu})^{1/2}=$$

Sharpe du marché

$X,Y\in\mathcal{L}^2$

alors $\exists\, a,b$ et une v.a ε tels que $Y=a+bX+\varepsilon$

avec $E\varepsilon=0$ et $Cov(\varepsilon,X)=0$

$$b=\frac{Cov(X,Y)}{Var\,X},\quad a=E(Y)-E[X]$$

$$\varepsilon=Y-a-bX$$

Slide 25.

$z=Y-(1+r)e$

$$z(a_m^*)=\frac{\tilde{\mu}'\Omega^{-1}z}{\tilde{\mu}'\Omega^{-1}e}$$

$$E(z(a_m^*))=\frac{\tilde{\mu}'\Omega^{-1}\tilde{\mu}}{\tilde{\mu}'\Omega^{-1}e}$$

Si $Var\,Z=\Omega$ alors $Var(a'Z)=a'\Omega a$

Ptf de marché efficient

$Z_i=\alpha_i+\beta_i Z(a_m^*)+\varepsilon_i \quad \forall i$

$\rightarrow$ $\alpha=E[Z(a_m^*)]-bE[Z(a_m^*)]$

$$\alpha=\tilde{\mu}-b\frac{\tilde{\mu}\Omega^{-1}\tilde{\mu}}{\tilde{\mu}\Omega^{-1}e}$$

$\rightarrow$ $$\beta=\frac{Cov(X,Y)}{Var(X)}=\frac{Cov(Z(a_m^*),Z)}{Var(Z(a_m^*))}=\frac{Cov(\tilde{\mu}\Omega^{-1}Z,Z)}{\tilde{\mu}\Omega^{-1}e\,Var(Z(a_m^*))}$$

$$Var(Z(a_m^*))=Var\left(\frac{\tilde{\mu}'\Omega^{-1}Z}{\tilde{\mu}'\Omega^{-1}e}\right)=\frac{(\tilde{\mu}'\Omega^{-1})\Omega(\Omega^{-1}\tilde{\mu})}{(\tilde{\mu}'\Omega^{-1}e)^2}$$

- $$Var(Z(a_m^*))=\frac{\tilde{\mu}'\Omega^{-1}\tilde{\mu}}{(\tilde{\mu}'\Omega^{-1}e)^2}$$

- $$Cov(\tilde{\mu}\Omega^{-1}Z,Z)=\tilde{\mu}\Omega^{-1}Var(Z)=\tilde{\mu}\Omega^{-1}\Omega=\tilde{\mu}$$

$$\beta=\frac{\tilde{\mu}\Omega^{-1}e\tilde{\mu}}{\tilde{\mu}\Omega^{-1}\tilde{\mu}}\Rightarrow\boxed{\alpha=0}$$

Markowitz

- $y_i = \frac{p_{i,1}}{p_{i,0}}$ rendement $Y = (y_1, \dots, y_N)'$. $E(Y) = \mu \in \mathbb{R}^N$. $Var(Y) = \Omega$
- Valeur en t $\quad V_t = a_0 p_t^{ASR} + a' p_t \quad \to a = (a_1, \dots, a_N)'$ et $p_t = (p_{1,t}, \dots, p_{N,t})'$
- $V_0 = a_0 + a' p_0$ $\quad$ • $V_1 = a_0(1+r) + a' \mathrm{diag}(p_0) Y$
- $w_a = \mathrm{diag}(p_0) a$ $\quad \boxed{E(V_1) = a_0(1+r) + a'\mathrm{diag}(p_0)\mu} \quad \boxed{Var(V_1) = a'\mathrm{diag}(p_0)\Omega\,\mathrm{diag}(p_0) a}$

- Problème d'optimisation à Résoudre:

$$(E') \to \begin{cases} \max\limits_{(a_0,a)} & a_0(1+r) + a'\mathrm{diag}(p_0)\mu \\ \text{S.c} & a'\mathrm{diag}(p_0)\Omega\,\mathrm{diag}(p_0)a = \sigma^2 \text{ et } \boxed{a_0 + a'p_0 = v} \end{cases}$$

On pose $\underline{v = a_0 + w_a' e}$ avec $e = (1, \dots, 1)'$

$$\Leftrightarrow (E') \begin{cases} \max\limits_{w_a} & (v - w_a' e)(1+r) + w_a' \mu \\ \text{Sc} & w_a' \Omega w_a = \sigma^2 \end{cases} \qquad \bullet \begin{cases} v = a_0 + w_a' e \\ \sigma^2 = w_a' \Omega w_a \end{cases}$$

On pose $\underline{\tilde{\mu} = \mu - (1+r)e}$ enlève car dep pas de $v(1+r)$

$$\Leftrightarrow (E') \begin{cases} \max\limits_{w_a} & v(1+r) + w_a'\tilde{\mu} \\ & \text{sc } w_a'\Omega w_a - \sigma^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{\mu} - \lambda \Omega w_a = 0 \\ w_a' \Omega w_a - \sigma^2 = 0 \end{cases} \Rightarrow \begin{cases} w_a = \frac{1}{\lambda}\Omega^{-1}\tilde{\mu} \\ a = \frac{1}{\lambda}(\mathrm{diag}(p_0))^{-1}\Omega^{-1}\tilde{\mu} \\ a_0 = v - \frac{1}{\lambda}\tilde{\mu}'\Omega^{-1}e \\ \lambda = \frac{1}{\sigma}(\tilde{\mu}'\Omega^{-1}\tilde{\mu})^{1/2} \end{cases}$$

- $\boxed{w_a = \frac{1}{\lambda}\Omega^{-1}\tilde{\mu}} \to \boxed{a = \frac{1}{\lambda}(\mathrm{diag}(p_0))^{-1}\Omega^{-1}\tilde{\mu}}$

$\boxed{a_0 = v - \frac{1}{\lambda}\tilde{\mu}'\Omega^{-1}e}$ $\quad \boxed{\lambda = \frac{1}{\sigma}(\tilde{\mu}'\Omega^{-1}\tilde{\mu})^{1/2}}$

- $\boxed{E[V_1] = v(1+r) + \frac{1}{\lambda}\tilde{\mu}'\Omega^{-1}\tilde{\mu}}$ $\quad \boxed{Var(V_1) = \sigma = \frac{(\tilde{\mu}'\Omega^{-1}\tilde{\mu})^{1/2}}{\lambda}}$

- $\boxed{E[V_1] = v(1+r) + \sqrt{Var(V_1)}\,(\tilde{\mu}'\Omega^{-1}\tilde{\mu})^{1/2}}$

on écrit $\boxed{P = \alpha_0(\lambda) P_0 + \alpha^*(\lambda) P^*}$

- $P_0 = ASR$
- $\alpha_0(\lambda) = v - \frac{1}{\lambda} \tilde{\mu}' \Omega^{-1} e$
- $\alpha^*(\lambda) = \frac{1}{\lambda}$
- $P^* = (\mathrm{diag}(P_0))^{-1} \Omega^{-1} \tilde{\mu}$

Théorème de séparation en 2 fonds :

Tout portefeuille efficient est la combinaison linéaire entre P_0 (l'ASR) et P^a qui ne contient que des actifs risqués.

Sharpe $S(a) = \frac{E[V_1(a_0, a) - v(1+r)]}{\sqrt{Var[V_1(a_0, a)]}} = \frac{w_a' \tilde{\mu}}{(w_a' \Omega w_a)^{1/2}}$

$S = \frac{1}{\lambda} \frac{\tilde{\mu}' \Omega^{-1} \tilde{\mu}}{\sigma} = (\tilde{\mu}' \Omega^{-1} \tilde{\mu})^{1/2} = S$

CAPM

- on note $z = Y - (1+r)e$ vecteur rentabilité net

$z(\alpha^*) = \frac{\tilde{\mu}' \Omega^{-1} z}{\tilde{\mu} \Omega^{-1} e}$ pour un ptf efficient

on a $\underline{z = \beta z(\alpha_m^*) + \varepsilon}$ $\quad \beta = \frac{Cov(z, z(\alpha_m^a))}{Var(z(\alpha_m^*))}$ $\cdot\ E[\varepsilon] = 0$ $\cdot\ Cov[\varepsilon, z(a_m)] = 0$

Et : $E[z_i] = \beta_i E[z(a_m^*)]$

$Var[z_i] = \beta_i^2 Var[z(a_m)] + \sigma_i^2$ avec $\sigma_i^2 = E[\varepsilon_i^2]$

Risque idiosyncratique

on prend a un ptf tq : $z(a) = w_a' z = \sum_i w_{a,i} z_i = \sum_i w_{a,i}(\beta_i z(a_m^*) + \varepsilon_i)$

on pose $\to \beta(a) = \sum_i w_{a,i} \beta_i \quad \varepsilon(a) = \sum_i w_{a,i} \varepsilon_i = w_a' \varepsilon$

$\hookrightarrow E[z(a)] = \beta(a) E[z(a_m^*)]$

$\hookrightarrow V[z(a)] = \beta(a)^2 Var[z(a_m^a)] + Var[w_a' \varepsilon]$

$= \beta(a)^2 Var[z(a_m^a)] + w_a' Var[\varepsilon] w_a$

on pose $\bar{w}_a = \frac{w_a}{\underbrace{w_a e}_{1}}$

$V[z(a)] = \beta(a)^2 Var[z(a_m^*)] + \bar{w}_a' Var[\varepsilon] \bar{w}_a$